TWO-LAYER FILM FLOW OVER THE SURFACE OF A ROTATING DISC

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Results are presented of numerical computations of a stationary axisymmetric film flow consisting of two immiscible fluids.

Spreading of a liquid film over a rotating plane surface is encountered in many technological processes whose analysis requires knowledge of the hydrodynamic characteristics of such a flow. A single-layer flow is examined in a number of papers whose survey is presented in [1]. The collocation method [1, 2] is used in this paper to analyze a two-layer flow.

Let viscous incompressible fluids be delivered at the constant mass flow rates Q_1 and Q_2 near the axis of rotation of a disc, where the subscript 1 corresponds to the lower fluid, and 2 to the upper. Analogously to [2], the velocity components in a fixed cylindrical coordinate system connected to the center of disc rotation are represented in the form

$$u_r = \omega r \delta^2 u,\ u_\theta = \omega r (1 + \delta^2 v),\ u_z = \omega H_* \delta^2 w.$$

Without taking account of the surface tension on the interfaces the system of equations and boundary conditions describing the stationary axisymmetric film flow has the following form [2] to the accuracy of terms of order $(H_*/r)^2$

$$\frac{\partial u}{\partial x} + 2u + \frac{\partial w}{\partial y} = 0, \tag{1}$$

$$\alpha_f \frac{\partial^2 u}{\partial y^2} + 1 + 2\delta^2 v - \delta^4 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial y} + u^2 - v^2 \right) = 0, \tag{2}$$

$$\alpha_f \frac{\partial^2 v}{\partial y^2} - 2\delta^2 u - \delta^4 \left(u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial y} + 2uv \right) = 0, \tag{3}$$

$$y = 0:\ u = v = w = 0, \tag{4}$$

$$y = H_1:\ u \frac{dH_1}{\partial x} = w,\quad [u] = [v] = \left[\rho\nu \frac{\partial u}{\partial y} \right] = \left[\rho\nu \frac{\partial v}{\partial y} \right] = 0, \qquad (4)$$

$$y = H_2:\ u \frac{dH_2}{dx} = w,\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0, \tag{6}$$

where H_1, H_2 are thicknesses of the lower layer and the film, respectively, $x = \ln (r/R)$, $y = z/H_*$, the notation $[a] = a_2 - a_1$ is used in (5), the quantities a_1, a_2 refer to the lower and upper layers, respectively. Here (1) is the continuity equation, (2) and (3) are the equations of motion for the radial and azimuthal velocity components, respectively, where $\alpha_f = 1$ for the lower layer and $\alpha_f = \alpha$ for the upper, (4) is the adhesion and nonpenetration condition on the disc surface, (5) is the kinematic condition, the conditions of continuity of the velocity and tangential stress components on the boundary between the layers, and (6) is the kinematic condition and conditions for no tangential stresses on the free surface.

Spreading of the film is considered as a Cauchy problem with the initial conditions formulated below for x = 0.

The streamlines $y = h_n(x)$ and the values of the velocity components on them $u_n(x) = u(x, h_n(x))$, $v_n(x) = v(x, h_n(x))$, n = 1, 2, ..., N are introduced from the numerical solution, where $h_M \equiv H_1$ and $h_N \equiv H_2$. A system of ordinary differential equations can be obtained for the variables h_n, u_n, v_n from (1)-(6) [1]

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$$\frac{dh_n}{dx} = \frac{dh_{n-1}}{dx} - (h_n - h_{n-1})\left[\frac{1}{u_n + u_{n-1}}\left(\frac{du_n}{dx} + \frac{du_{n-1}}{dx}\right) + 2\right],$$

$$\frac{du_n}{dx} = \frac{1}{u_n}\left[v_n\left(v_n + \frac{2}{\delta^2}\right) + \frac{1}{\delta^4}\left(\alpha_f \frac{\partial^2 u}{\partial y^2}\bigg|_{y=h_n} + 1\right)\right] - u_n, \tag{7}$$

$$\frac{dv_n}{dx} = \frac{\alpha_f}{\delta^4 u_n}\frac{\partial^2 v}{\partial y^2}\bigg|_{y=h_n} - 2\left(v_n + \frac{1}{\delta^2}\right), \quad n = 1, 2, \ldots, N, \; h_0 \equiv 0, \; u_0 \equiv 0,$$

where $\alpha_f = 1$ for $n = 1, 2, \ldots, M$ and $\alpha_f = \alpha$ for $n = M + 1, \ldots, N$.

A tau-approximation using displaced Chebyshev polynomials of the first kind defined by the formulas [4]

$$\varphi_1 = 1, \; \varphi_2 = 2\eta - 1, \; \varphi_k = 2\varphi_2\varphi_{k-1} - \varphi_{k-2}, \; k = 3, 4, \ldots$$

is applied for the calculation of the second derivatives in the right sides of (7). Two approximate formulas

$$U_1 = \sum_{k=1}^{M+2} a_k \varphi_k\left(\frac{y}{h_M}\right), \; U_2 = \sum_{k=1}^{N-M+3} a_{M+2+k}\varphi_k\left(\frac{y - h_M}{h_N - h_M}\right),$$

are constructed here for the velocity components, for example u, whose expansion coefficients a_k, $k = 1, 2, \ldots, N + 5$ are solutions of systems of linear algebraic equations

$$\sum_{k=1}^{M+2} a_k\varphi_k\left(\frac{h_n}{h_M}\right) = u_n, \; n = 1, 2, \ldots, M,$$

$$\sum_{k=1}^{N-M+3} a_{M+2+k}\varphi_k\left(\frac{h_n - h_M}{h_N - h_M}\right) = u_n, \; n = M, \ldots, N, \tag{8}$$

$$\sum_{k=1}^{M+2} a_k\varphi_k(0) = 0, \quad \sum_{k=1}^{N-M+3} a_{M+2+k}\varphi_k'(1) = 0, \tag{9}$$

$$\frac{1}{h_M}\sum_{k=1}^{M+2} a_k\varphi_k'(1) - \frac{\alpha\lambda}{h_N - h_M}\sum_{k=1}^{N-M+3} a_{M+2+k}\varphi_k'(0) = 0, \tag{10}$$

$$\frac{1}{h_M^2}\sum_{k=1}^{M+2} a_k\varphi_k''(1) - \frac{\alpha}{(h_N - h_M)^2}\sum_{k=1}^{N-M+3} a_{M+2+k}\varphi_k''(0) = 0, \tag{11}$$

where (8) is the condition for equality of the functions U_1 and U_2 to values of the velocity components of u on the streamlines, (9) is the approximation of the boundary conditions on the disc and the film surface, (10) and (11) are approximation of the tangential stress continuity conditions and the quantity $\nu\partial^2 u/\partial y^2$ on the boundary between the layers, the last relationship follows from the equations of motion and continuity of the velocity on the boundary.

After determining a_k, $k = 1, 2, \ldots, N + 5$, the expansion coefficients can be evaluated a Chebyshev polynomial series of the functions d^2U_1/dy^2 and d^2U_2/dy^2 and the value of $\partial^2 u/\partial y^2$ can later be determined on the streamlines [1].

The initial conditions

$$h_n(0) = \frac{H_1(0)\, n}{M}, \; n = 1, \ldots, M, \; h_n(0) = H_1(0) + \frac{[1 - H_1(0)](n - M)}{N - M},$$

$$n = M + 1, \ldots, N, \; u_n(0) = U(h_n), \; v_n(0) = V(h_n), \; n = 1, \ldots, N,$$

are appended to (7), where U, V, are given functions. The value of the film thickness at x = 0 is considered as the characteristic scale.

Integration of (7) is by the Adams–Bashforth method of second-order accuracy [3].

There follows from (1), (4)-(6) and the axisymmetry of the flow that

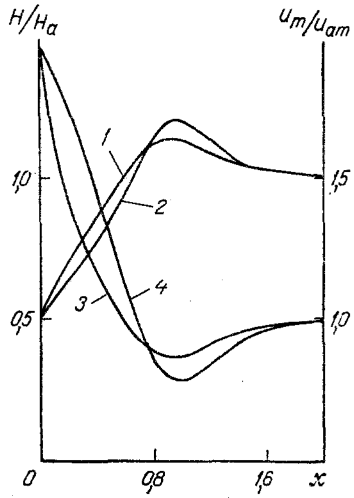

Fig. 1. Dependence of the thicknesses and mean radial velocities on the radius for the initial profile (13) for $\alpha = \lambda = 0.5$, $\delta = 1$, $q_1(0) = 0.6$, $q_2(0) = 1.5$: 1) $H_1/H_{1\alpha}$; 2) $H_2/H_{2\alpha}$; 3) $u_{m1}/u_{\alpha m1}$; 4) $u_{m2}/u_{\alpha m2}$; M = 5, N = 10.

$$q_1(x) \equiv \int_0^{H_1} u\,dy = \frac{Q_1\nu_1}{2\pi r^2\omega^2 H_*^3},\quad q_2(x) \equiv \int_{H_1}^{H_2} u\,dy = \frac{Q_2\nu_1}{2\pi r^2\omega^2 H_*^3}.$$

For small values of the parameter δ the problem (1)-(6) without initial conditions has a solution whose principal terms of the expansion in the quantity δ^4 have the form

$$\begin{gathered}
0 \leqslant y \leqslant H_{1a}:\ u_a = -\frac{1}{2}y^2 + [(1-\lambda)H_{1a} + \lambda H_{2a}]\,y,\\
v_a = \delta^2\left\{-\frac{1}{12}y^4 + \frac{1}{3}[(1-\lambda)H_{1a} + \lambda H_{2a}]\,y^3 +\right.\\
\left.+\,2\left[\left(\frac{\lambda}{3\alpha} + \lambda - \lambda^2 - \frac{1}{3}\right)H_{1a}^3 + \lambda\left(2\lambda - 1 - \frac{1}{\alpha}\right)H_{1a}^2 H_{2a} + \lambda\left(\frac{1}{\alpha} - \lambda\right)H_{1a}H_{2a}^2 - \frac{\lambda}{3\alpha}H_{2a}^3\right]y\right\},\\
H_{1a} \leqslant y \leqslant H_{2a}:\ u_a = -\frac{1}{2\alpha}y^2 + \frac{H_{2a}}{\alpha}y +\\
+\left(\frac{1}{2} - \lambda + \frac{1}{2\alpha}\right)H_{1a}^2 + \left(\lambda - \frac{1}{\alpha}\right)H_{1a}H_{2a},\\
v_a = \delta^2\left\{-\frac{1}{12\alpha^2}y^4 + \frac{H_{2a}}{3\alpha^2}y^3 + \frac{H_{1a}}{2\alpha}\left[\left(1 - 2\lambda + \frac{1}{\alpha}\right)H_{1a} +\right.\right.\\
\left.+\,2\left(\lambda - \frac{1}{\alpha}\right)H_{2a}\right]y^2 + \frac{H_{2a}}{\alpha}\left[-\frac{2}{3\alpha}H_{2a}^2 + 2\left(\frac{1}{\alpha} - \lambda\right)H_{1a}H_{2a} +\right.\\
\left.+\left(2\lambda - 1 - \frac{1}{\alpha}\right)H_{1a}^2\right]y + \left(\frac{5\lambda}{3} - \frac{5}{12} + \frac{5\lambda}{3\alpha} - 2\lambda^2 -\right.\\
\left.-\frac{5}{12\alpha^2} - \frac{1}{2\alpha}\right)H_{1a}^4 + \left(4\lambda^2 - \frac{5\lambda}{3} - \frac{5\lambda}{\alpha} + \frac{5}{3\alpha^2} + \frac{1}{\alpha}\right)H_{1a}^3 H_{2a} +\\
\left.+\,2\left(\frac{2\lambda}{\alpha} - \lambda^2 - \frac{1}{\alpha^2}\right)H_{1a}^2 H_{2a}^2 + \frac{2}{3\alpha}\left(\frac{1}{\alpha} - \lambda\right)H_{1a}H_{2a}^3\right\},
\end{gathered}\tag{12}$$

the values of $H_{1\alpha}$ and $H_{2\alpha}$ are determined from the algebraic equations

$$H_{2a} = \frac{2q_1}{\lambda H_{1a}^2} + \left(1 - \frac{2}{3\lambda}\right)H_{1a},\quad q_2 + \frac{1}{6\alpha}\left(H_{2a}^3 - H_{1a}^3\right) - \frac{H_{2a}}{2\alpha}\left(H_{2a}^2 - H_{1a}^2\right) -$$

$$-\left[\left(\frac{1}{2} - \lambda + \frac{1}{2\alpha}\right)H_{1a} + \left(\lambda - \frac{1}{\alpha}\right)H_{2a}\right]H_{1a}(H_{2a} - H_{1a}) = 0.$$

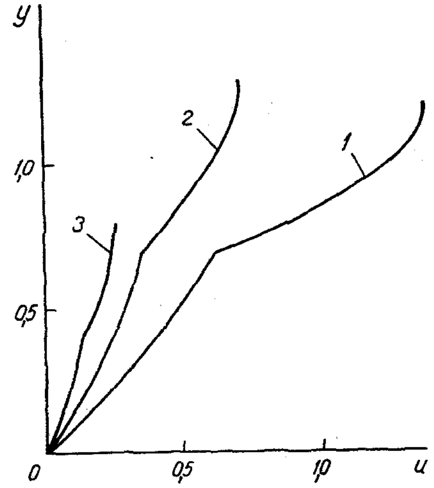

Fig. 2. Radial flow velocity: 1) x = 0.48; 2) 0.72; 3) 1.44.

Results of the computations show that, independently of the form of the initial conditions, the solution after the formation section takes on the form (12). The length x of the formation section and the degree of difference between the solutions on it from (12) depend on the parameters δ, λ, α, the flow rates $q_1(0)$, $q_2(0)$, and the initial velocity component profiles. The case

$$0 \leqslant y \leqslant H_1\colon\ U = \alpha b\{2[\lambda + (1-\lambda)H_1]y - y^2\},\ V = 0,$$
$$H_1 \leqslant y \leqslant 1\colon\ U = b\{2y - y^2 + [1 + \alpha(1-2\lambda)]H_1^2 + 2(\alpha\lambda - 1)H_1\},\ V = 0, \tag{13}$$

was examined when studying the dependence of x_α on the parameters, where the parameters b, H_1 are related to $q_1(0)$, $q_2(0)$. The examples presented below correspond to values of the parameters for which the quantity x_α is of the order of 1. For α = λ = 0.5, δ = 1, $q_2(0) = 2.5q_1(0)$ the values of x_α = 1.44, 1.76, 2.0 for $q_1(0)$ = 0.2, 0.4, 0.6; in the case $q_1(0)$ = 0.2, $q_2(0)$ = 0.5, α = λ = 0.5 for δ^2 = 0.2; 0.6; 1.4; then x_α = 0.2, 1.04, 1.68; if $q_1(0)$ = 0.2, $q_2(0)$ = 0.5, α = λ = 2, then x_α = 0.28, 1.04, 1.44 for δ^2 = 0.2, 0.6, 1.0. The condition $(|1 - H_1/H_{1a}| + |1 - H_2/H_{2a}| + |1 - u_{m1}/u_{am1}| + |1 - u_{m2}/u_{am2}|) < 0{,}04$, is used as the criterion for the selection of x_α, where $u_{m1} = q_1/H_1$, $u_{m2} = q_2(H_2 - H_1)$ are the radial velocities; u_{am1}, u_{am2} are the corresponding asymptotic values.

The influence of the initial azimuthal velocity was examined for α = λ = 0.5, δ = 1, $q_1(0)$ = 0.4, $q_2(0)$ = 1, where the radial velocity profile had the form (13) and V = 0, −0.2U, −0.4U, and x_α = 1.76 in all cases.

Shown in Fig. 1 are the dependences H_1, H_2, u_{m1}, u_{m2} in one of the computations for which examples of the radial velocity profiles are represented in Fig. 2.

Therefore, the numerical method examined, which is an extension of that proposed in [1], permits computation of the flows of two-layer films of relatively large thickness for which the length of the formation section of the asymptotic solution is commensurate with the disc radius and can be used to study flows in which heat and mass transfer exists between the fluids.

NOTATION

Q_1, Q_2, fluid flow rates, ω, angular velocity of the disc; ρ_1, ρ_2, ν_1, ν_2, fluid densities and kinematic viscosity coefficients; H_*, film characteristic thickness; R, least flow domain radius, $\delta = H_*\sqrt{\omega/\nu_1}$, $\alpha = \nu_2/\nu_1$, $\lambda = \rho_2/\rho_1$, parameters; r, θ, z, and u_r, u_θ, u_z, cylindrical coordinate system and the velocity components, and x_α, dimensionless length of the section of asymptotic solution formation.

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SHAPES OF ANNULAR LAYERS OF FLUID ON THE SURFACE OF A ROTATING CYLINDER

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A qualitative and quantitative study is made of the equilibrium forms of plane and axisymmetric fluid layers.

The power-engineering, chemical, and building sectors make use of production processes based on the phenomenon of instability of the free surface of a layer of fluid. For example, the production of thermal insulating wool by the centrifugal-roller method involves the disintegration of a layer formed on the surface of a rapidly rotating cylinder when a mineral melt falls onto the roller [1]. The study [2] presented photographs reflecting the stages of formation of layers of a viscous fluid (such as glycerin or aqueous solutions of glycerin) obtained on an experimental unit which included a rotating cylindrical roller mounted on the horizontal shaft of an electric motor. Some of the liquid which falls onto the roller is thrown off by centrifugal forces. The rest of the liquid is entrained by the rotating surface in the form of an annular layer, with drops separating from the layer about its entire perimeter. When a certain period of time has elapsed after cessation of the supply of fluid, a steady-state regime is established in which the fluid ring, with a smooth surface, rotates as a solid. With an increase in the speed of rotation, the surface of the ring may acquire a wavy shape — as in the photograph shown in Fig. When the speed is increased above a certain critical value, more of the mass of the fluid is thrown from the roll and another stationary fluid ring with a wavy free surface is established.

The studies [3-7] used the small parameter method to theoretically investigate the forms of equilibrium of liquid streams and layers near bifurcation points. Here, we perform a quantitative and qualitative study of nonlinear solutions in relation to values of the characteristic parameters.

1. Formulation of the Problem and Derivation of the Basic Equation. We introduce a cylindrical coordinate system 0, x, y, φ (Fig. 2). The motion of the viscous fluid is described by the Navier–Stokes equations, the continuity equation, and the equation of the free surface:

$$\frac{d\mathbf{u}}{dt} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u},\ \nabla\mathbf{u} = 0,\ \rho = \mathrm{const}, \tag{1}$$

$$\frac{dh}{dt} = v,\ y = h(x,\ \varphi,\ t). \tag{2}$$

The normal and shear stresses on the external surface of the layer satisfy the conditions in [8]. Due to adhesion, the components of velocity on the roller surface have the following values:

$$u = 0,\ v = 0,\ w = 1,\ y = 1. \tag{3}$$

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